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The Modulus of Continuity of the Remainder in the Approximation of Lipschitz Functions

WALTER R. BLOOM

School of Mathematical and Physical Sciences, Murdoch University, Murdoch, Western Australia 6153, Australia

AND

DAVID ELLIOTT

Department of Mathematics, University of Tasmania, GPO Box 252C, Hobart, Tasmania 7001, Australia

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1. INTRODUCTION

Let X be a normed linear space of complex-valued functions defined on $E \subset R^m$, with norm denoted by $\|\cdot\|_X$. We shall assume that there exists $h_0 > 0$ and $\omega: X \times [0, h_0] \to [0, \infty)$ with the following properties:

$$\omega(f_1 + f_2, h) \leqslant \omega(f_1, h) + \omega(f_2, h), \tag{1.1}$$

$$\omega(f, h) \leqslant 2 \|f\|_X. \tag{1.2}$$

In particular cases ω will be a modulus of continuity and hence we shall use this terminology in general.

Suppose that X admits an increasing dense sequence $\{S_n\}$ of linear subspaces satisfying the following Bernstein-type inequality; there exist positive constants λ, A such that for each n and $s_n \in S_n$,

$$\omega(s_n, h) \leqslant A n^\lambda \|s_n\|_X h \tag{1.3}$$

for all $h \in [0, h_0]$. Examples of such spaces will be given in Section 3.

For $f \in X$ choose a sequence $\{s_n\}$, where $s_n \in S_n$, such that $\lim_{n\to\infty} \|f - s_n\|_X = 0$. It is the purpose of this paper to discuss the modulus of continuity of the remainder $r_n = f - s_n$, $n = 1, 2,\ldots$. This problem has been considered previously in some special cases (see Section 3), but not in the generality given here. Our principal result is presented in the next section,

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giving an estimate for $\omega(r_n, h)$. It is also shown that this estimate cannot be improved (see Section 3.1).

2. An Upper Bound for $\omega(r_n, h)$

For the spaces X, S_n satisfying the conditions given in the Introduction we have the following theorem.

Theorem 1. *Let $f \in X$, $s_n \in S_n$ and suppose there is a sequence $\{u(n)\}$, where $\|r_n\|_X = \|f - s_n\|_X \leqslant u(n)$ for $n = 1, 2,\ldots$, satisfying the following property; there exist $M > 1$, $\beta \in (0, 1)$ and n_0 such that for all $n \geqslant n_0$ and $k = 1, 2,\ldots$,*

$$u(M^k n) \leqslant M^{-k\lambda\beta} u(n). \tag{2.1}$$

Then, for $0 \leqslant h \leqslant h_0$,

$$\omega(r_n, h) \leqslant B n^{\lambda\beta} u(n)\, h^{\beta}, \tag{2.2}$$

where B is independent of both n and h.

Proof. Choose $n \geqslant n_0$. From (2.1) it follows that $\lim_{k\to\infty} u(M^k n) = 0$, so that $\lim_{k\to\infty} \|f - s_{M^k n}\|_X = 0$. If, for $k = 1, 2,\ldots$, we define

$$t_{k,n} = s_{M^k n} - s_{M^{k-1} n}, \tag{2.3}$$

then we can write

$$r_n = f - s_n = \sum_{k=1}^{\infty} t_{k,n}.$$

From (2.3), (2.1),

$$\begin{aligned} \|t_{k,n}\|_X &\leqslant \|f - s_{M^k n}\|_X + \|f - s_{M^{k-1} n}\|_X \\ &\leqslant u(M^k n) + u(M^{k-1} n) \\ &\leqslant (1 + M^{\lambda\beta})\, M^{-k\lambda\beta} u(n). \end{aligned} \tag{2.4}$$

For any non-negative integer ν we write

$$r_n = \sum_{k=1}^{\nu} t_{k,n} + \sum_{k=\nu+1}^{\infty} t_{k,n} = R_1 + R_2$$

say, where we define R_1 to be zero if $\nu = 0$. We shall estimate the modulus of continuity of each of these sums.

Since $t_{k,n} \in S_{M^k n}$ we have, using (1.1) and (1.3),

$$\omega(R_1, h) \leqslant \sum_{k=1}^{\nu} \omega(t_{k,n}, h) \leqslant Ah \sum_{k=1}^{\nu} (M^k n)^\lambda \|t_{k,n}\|_X.$$

Applying (2.4) gives

$$\begin{aligned}\omega(R_1, h) &\leqslant Ahn^\lambda(1 + M^{\lambda\beta})\, u(n) \sum_{k=1}^{\nu} M^{k\lambda(1-\beta)} \\ &\leqslant Ahn^\lambda(1 + M^{\lambda\beta})\, u(n)\, M^{\lambda(1-\beta)(1+\nu)}/(M^{\lambda(1-\beta)} - 1),\end{aligned} \tag{2.5}$$

where we have made use of the conditions $M > 1$ and $\beta < 1$. Now assume that $0 < h \leqslant n^{-\lambda}$, and choose ν such that

$$(M^{\nu+1} n)^{-\lambda} < h \leqslant (M^\nu n)^{-\lambda}. \tag{2.6}$$

Then, from (2.5) and (2.6), we obtain

$$\omega(R_1, h) \leqslant \{AM^\lambda(1 + M^{-\lambda\beta})/(M^{\lambda(1-\beta)} - 1)\}\, n^{\lambda\beta} u(n)\, h^\beta. \tag{2.7}$$

To estimate $\omega(R_2, h)$ we use (1.2) to obtain

$$\omega(R_2, h) \leqslant 2\, \|R_2\|_X \leqslant 2 \sum_{k=\nu+1}^{\infty} \|t_{k,n}\|_X.$$

Then, as for the previous estimate, (2.4) gives

$$\begin{aligned}\omega(R_2, h) &\leqslant 2(1 + M^{\lambda\beta})\, u(n) \sum_{k=\nu+1}^{\infty} M^{-k\lambda\beta} \\ &= 2(1 + M^{\lambda\beta})\, u(n)\, M^{-\lambda\beta\nu}/(M^{\lambda\beta} - 1),\end{aligned}$$

where we have made use of the conditions $M > 1$ and $\beta > 0$. It follows from (2.6) that

$$\omega(R_2, h) \leqslant \{2(1 + M^{\lambda\beta})\, M^{\lambda\beta}/(M^{\lambda\beta} - 1)\}\, n^{\lambda\beta} u(n)\, h^\beta. \tag{2.8}$$

Combining (2.7) and (2.8) we have that for all $h \leqslant n^{-\lambda}$,

$$\omega(r_n, h) \leqslant Bn^{\lambda\beta} u(n)\, h^\beta,$$

where

$$B = AM^\lambda(1 + M^{-\lambda\beta})/(M^{\lambda(1-\beta)} - 1) + 2(1 + M^{\lambda\beta})\, M^{\lambda\beta}/(M^{\lambda\beta} - 1). \tag{2.9}$$

To complete the proof it remains only to consider those values of h for which $h > n^{-\lambda}$. Again from (1.2) and (2.4),

$$\begin{aligned}\omega(r_n, h) &\leqslant 2 \sum_{k=1}^{\infty} \|t_{k,n}\|_X \\ &\leqslant 2(1+M^{\lambda\beta})\, u(n)/(M^{\lambda\beta}-1) \\ &\leqslant \{2(1+M^{\lambda\beta})/(M^{\lambda\beta}-1)\}\, n^{\lambda\beta} u(n)\, h^{\beta} \\ &\leqslant B n^{\lambda\beta} u(n)\, h^{\beta}.\end{aligned}$$

Thus the theorem is established with B defined by (2.9).

Of particular importance is the set of those functions in X satisfying a Lipschitz condition. We say that $f \in X$ satisfies a Lipschitz condition of order α, $\alpha > 0$, if there exist $M > 0$ and $0 < h' \leqslant h_0$ such that $\omega(f, h) \leqslant M h^{\alpha}$ for all $h \leqslant h'$. In this case the infimum of all such M is called the Lipschitz constant of f, and the set of all such functions will be denoted by $\mathrm{Lip}(X; \alpha)$. As an immediate corollary to Theorem 1 we have

COROLLARY 1. *Under the conditions of Theorem* 1, *f and the remainders r_n belong to* $\mathrm{Lip}(X; \beta)$.

Proof. From (2.2) we see that $r_{n_0} \in \mathrm{Lip}(X; \beta)$, and since $s_{n_0} \in \mathrm{Lip}(X; 1) \subset \mathrm{Lip}(X; \beta)$ by (1.3), it follows that $f = s_{n_0} + r_{n_0} \in \mathrm{Lip}(X; \beta)$. Now use the equality $r_n = f - s_n$ to deduce that $r_n \in \mathrm{Lip}(X; \beta)$ for all $n = 1, 2, \ldots$.

From (1.3) we see that, although $s_n \in \mathrm{Lip}(X; 1)$, the coefficient of h depends upon n and, in general, this will be unbounded as $n \to \infty$. However, by imposing a suitable smoothness condition on f, together with a certain rate of convergence on $\{u(n)\}$, we can deduce that $s_n \in \mathrm{Lip}(X; \sigma)$ for some $\sigma \in (0, 1)$, where the Lipschitz constants of the s_n are uniformly bounded.

COROLLARY 2. *Let $f \in \mathrm{Lip}(X; \mu)$ for some $\mu \in (0, 1)$ and suppose there exist $\{s_n\}$, $K > 0$ and $0 < v < \lambda\mu$ such that $u(n) = K n^{-v}$ for all $n = 1, 2, \ldots$. Writing $\sigma = \min(v/\lambda, \mu)$ we have*

$$\omega(s_n, h) \leqslant C h^{\sigma}, \tag{2.10}$$

where C is independent of both n and h.

Proof. In the notation of Theorem 1 choose $\beta = v/\lambda$; then $\beta \in (0, 1)$, (2.1) is trivially satisfied and, from (2.2), $\omega(r_n, h) \leqslant BK h^{v/\lambda}$, where BK is independent of both n and h. Now $s_n = f - r_n$ and, using (1.1), the result follows.

3. Some Applications

We shall consider three important applications of Theorem 1. Our first example also shows that, except for the constant B, inequality (2.2) cannot be improved.

3.1. *Uniform Approximation by Trigonometric Polynomials*

Take $m = 1$, $E = (-\infty, \infty)$ and $X = C_{2\pi}$, the space of continuous 2π-periodic functions with the uniform norm $(\|\cdot\|_\infty)$. The modulus of continuity of $f \in C_{2\pi}$ is given by $\omega(f, h) = \sup\{\|\tau_u f - f\|_\infty : |u| \leqslant h\}$, where $\tau_u f : x \to f(x+u)$ is just the u-translate of f. For each n, S_n will be the subspace of $C_{2\pi}$ spanned by $\{1, \cos x, ..., \cos nx; \sin x, ..., \sin nx\}$, so that each $s_n \in S_n$ will be a trigonometric polynomial of degree at most n. Now Bernstein's inequality (see [5, Theorem 47]) states that $\|s_n'\|_\infty \leqslant n \|s_n\|_\infty$, from which it follows that

$$\omega(s_n, h) \leqslant n \|s_n\|_\infty h; \tag{3.1}$$

on comparison with (1.3) we have $A = 1$ and $\lambda = 1$.

Now let $E_n(f) = \min\{\|f - s_n\|_\infty : s_n \in S_n\}$ denote the best approximation to f by members of S_n. If we choose $u(n) = Kn^{-\beta}$ then (2.1) is satisfied (for any $M > 1$) and (2.2) and Corollary 1 give that $f \in \mathrm{Lip}(C_{2\pi}; \beta)$. Thus we have that $E_n(f) = O(n^{-\beta})$ implies that $f \in \mathrm{Lip}(C_{2\pi}; \beta)$, which is a theorem due to S. N. Bernstein (see [5, Theorem 48]). Furthermore Jackson's theorem (see [5, Theorem 38]) shows that the exponent β of h cannot be improved (that is, increased). Similar results hold with $C_{2\pi}$ replaced by $L^p_{2\pi}$, the space of pth integrable 2π-periodic functions with the usual norm.

There is also the question as to whether the exponent $\lambda\beta$ of n in (2.2) can be decreased. We show that it cannot by considering the following example in $C_{2\pi}$. Suppose we have $\omega(r_n, h) \leqslant Cn^\gamma u(n) h^\beta$ for some $\gamma > 0$, where C is independent of both n and h. By a result of G. G. Lorentz (see [4, Theorem 6]) the function f given by

$$f(x) = \sum_{k=1}^{\infty} 2^{-k\beta} \cos(2^k x) \tag{3.2}$$

belongs to $\mathrm{Lip}(C_{2\pi}; \beta)$, where $0 < \beta < 1$. For n satisfying $2^{m-1} \leqslant n < 2^m$ write $s_n(x) = \sum_{k=1}^{m-1} 2^{-k\beta} \cos(2^k x)$. Then $r_n(x) = \sum_{k=m}^{\infty} 2^{-k\beta} \cos(2^k x)$ and

$$\|r_n\|_\infty = \sum_{k=m}^{\infty} 2^{-k\beta} = 2^{-(m-1)/\beta}/(2^\beta - 1).$$

Thus we may choose $u(n) = 2^{\beta} n^{-\beta}/(2^{\beta} - 1)$ and (2.1) will be satisfied for any $M > 1$. Now for $m = 1, 2,...$ we have

$$\begin{aligned} 2^{-m\beta}(1 - \cos 2) &\leqslant \sum_{k=m}^{\infty} 2^{-k\beta}(1 - \cos 2^{k-m+1}) \\ &= r_{2m-1}(0) - r_{2m-1}(2^{-m+1}) \\ &\leqslant \omega(r_{2m-1}, 2^{-m+1}) \\ &\leqslant C2^{(m-1)\gamma} 2^{-(2m-3)\beta}/(2^{\beta} - 1), \end{aligned}$$

from which it follows that

$$1 - \cos 2 \leqslant \{C2^{3\beta - \gamma}/(2^{\beta} - 1)\}\, 2^{m(\gamma - \beta)}.$$

For this to hold for $m = 1, 2,...$ we must have $\gamma \geqslant \beta$, thus showing that the exponent of n in (2.2) is best possible.

3.2. *Uniform Approximation by Algebraic Polynomials*

Take $m = 1$, $E = [-1, 1]$ and $X = C([-1, 1])$, the space of continuous functions on $[-1, 1]$ with the uniform norm. The modulus of continuity of $f \in C([-1, 1])$ is given by $\omega(f, h) = \sup\{|f(x_1) - f(x_2)| : x_1, x_2 \in [-1, 1], |x_1 - x_2| \leqslant h\}$. For each n, S_n will be the space of all algebraic polynomials of degree at most n. It follows from an inequality of A. A. Markov (see [5, Section 5.7.3]) that

$$\omega(s_n, h) \leqslant n^2 \|s_n\|_{\infty} h; \tag{3.3}$$

on comparison with (1.3) we have $A = 1$ and $\lambda = 2$. Now choose f with rth derivative belonging to $\mathrm{Lip}(C([-1, 1]); \alpha)$ for some $\alpha \in (0, 1]$, and for each n let p_n denote the polynomial of degree at most n that best approximates f uniformly on $[-1, 1]$. By a result of D. Jackson (see [5, Theorem 45]),

$$\|f - p_n\|_{\infty} \leqslant An^{-r-\alpha},$$

where $A > 0$ is independent of n. Taking $u(n) = An^{-r-\alpha}$ we see that (2.1) is satisfied for all $M > 1$, provided we choose $0 < 2\beta < \min(r + \alpha, 2)$. Then, applying (2.2),

$$\omega(r_n, h) \leqslant Bn^{-(r+\alpha-2\beta)} h^{\beta}. \tag{3.4}$$

This result, in the case when $r = 0$, has previously been obtained by Kalandiya [3]. Kalandiya's lemma has been used [1] to prove convergence of quadrature rules for some Cauchy principal value integrals, and (3.4) could be useful in estimating the corresponding rates of convergence.

It is of some interest to apply Corollary 2 in this context. If we approximate to $f \in \mathrm{Lip}(C([-1,1]), \mu)$, $0 < \mu < 1$, by the sequence of its Bernstein polynomials $B_n(f)$ then, by Popoviciu's theorem (see, for example, [5, Section 5.7.7]), we can take $u(n) = Kn^{-\mu/2}$ so that $\sigma = \mu/4$ and, from (2.10), we have

$$\omega(B_n(f), h) \leqslant Ch^{\mu/4}. \tag{3.5}$$

In the particular case when $\mu = 1$, a direct argument based on properties of the Bernstein polynomials shows that $\omega(B_n(f), h) \leqslant Ch$, where C is independent of n. It is not known whether the exponent of h in (3.5) can be improved.

Finally, by applying an argument similar to that in Section 3.1, to the function $f(x) = \sum_{k=1}^{\infty} 2^{-2k\beta} T_{2^k}(x)$, where $0 < \beta < 1$ and $T_{2^k}(x) = \cos(2^k \arccos x)$, it can again be shown that the exponent of n in (2.2), which in this case is 2β, cannot be reduced. This result was overlooked by Kalandiya.

3.3. *Linear Spline Approximation*

As in Section 3.1 take $m = 1$, $E = (-\infty, \infty)$ and $X = C_{2\pi}$, but let S_n be the space of continuous 2π-periodic functions that are linear on each interval $[2(k-1)\pi/n, 2k\pi/n]$; $k = 1, 2, \dots, n$. Then for each $s_n \in S_n$,

$$\omega(s_n, h) \leqslant \pi^{-1} n \, \|s_n\|_\infty \, h; \tag{3.6}$$

on comparison with (1.3) we have $A = \pi^{-1}$ and $\lambda = 1$. For $f \in \mathrm{Lip}(C_{2\pi}; \alpha)$, $0 < \alpha \leqslant 1$, let p_n denote the piecewise linear function comprising the straight line segments connecting the points $(2(k-1)\pi/n, f(2(k-1)\pi/n))$ and $(2k\pi/n, f(2k\pi/n))$ and extended by periodicity to $(-\infty, \infty)$. Then, since in each interval $(2(k-1)\pi/n, 2k\pi/n)$ we have

$$\begin{aligned} r_n(x) = (n/2\pi)\{&(f(x) - f(2k\pi/n))(x - 2(k-1)\pi/n) \\ &+ (f(x) - f(2(k-1)\pi/n))(2k\pi/n - x)\}, \end{aligned}$$

it readily follows that $\|f - p_n\|_\infty \leqslant An^{-\alpha}$, and from Theorem 1 we have $\omega(r_n, h) \leqslant Bn^{-(\alpha-\beta)} h^\beta$ for $0 < \beta < 1$. This result has been used previously by B. G. Gabdulhaev [2] in the context of finding approximate solutions of singular integral equations with Hilbert kernel.

Note added in proof. Using the K-method in the theory of real interpolation it can be shown that the exponent of h in (3.5) can be taken to be μ. We are grateful to Dr. W. Dickmeis of Technische Hochschule, Aachen, for pointing this out to us.

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